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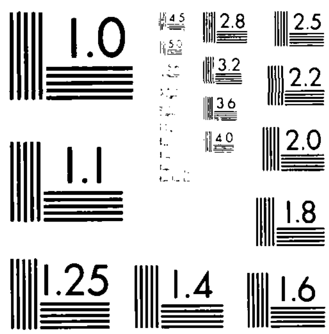

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A regenerative sampling plan is proposed for the sequential comparison of two populations having positive integral response. It is designed to be both an extension and an improvement of the play-the-winner rules for binary trials in the sense that a much wider variety of responses is allowed, the fraction of inferior selections approaches zero, and the play-the-winner rule is contained as a special case. Almost sure convergence and moment convergence in the pth order is studied for the fraction of inferior selections and for a maximum likelihood estimator of the mean response. A conditional test of hypothesis is given for the binary case.

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Abstract

A regenerative sampling plan is proposed for the sequential comparison of two populations having positive integral response. It is designed to be both an extension and an improvement of the play-the-winner rules for binary trials in the sense that a much wider variety of responses is allowed, the fraction of inferior selections approaches zero, and the play-the-winner rule is contained as a special case. Almost sure convergence and moment convergence in the pth order is studi d for the fraction of inferior selections and for a maximum likelihood estimator of the mean response. A conditional test of hypothesis is given for the binary case.

Section 1. Introduction

The problem of sequentially sampling two populations with unknown means so that the sum of observations is maximized has been formulated by Robbins (1952). For the binary case (success/failure trials), play-the-winner strategies have been shown to produce better results than a random selection of populations, in the sense that the fraction of inferior selections approaches the constant $q_A/(q_A+q_B)$ where $q_A < q_B$ are the failure probabilities (Robbins (1952), Zelen (1969), Wei and Durham (1978)). A randomized play-the-winner plan has been used for the assignment of patients to treatments in a controlled clinical study of a potentially life-saving medical procedure because of its tendency to put more patients on the better treatment (Bartlett et al (1985), Cornell et al (1986)).

The purpose of this paper is to present a sampling procedure in which the fraction of inferior selections approaches zero, in general, whenever the observed response is a positive integer-valued random variable. The main idea is to generate new samples on the two populations according to the cumulative response observed on each, as is done with the play-the-winner rules, but modified so that the sample sizes for the two populations are independent. The successive samples then correspond to the generations of two independent Galton-Watson branching processes and the attendant limit theory applies. Based on the observed successes on the two populations with a binary response, a conditional test of hypothesis is given along with explicit bounds on the power function. While other methods for dealing with the binary trials exist in which the fraction of inferior selections go to zero (Bather (1981)), they do not seem to have as tractable an inferential structure.

Section 2. Regenerative Sampling with a Positive Response

A sequence of stopping times called generation points are defined for the observations on each population independently of the observations on the other population. For the population $i = A,B$, let $R_1^i, R_2^i, R_3^i, \ldots$ be independent and identically distributed (i.i.d.) random variables taking positive integer values having a common mean, m_i. The R_k^i correspond to the observed responses on population i. Beginning with an initial sample of size u_i, a positive integer, the sequence T_n^i of generation points are defined by

$$(1) \qquad \begin{cases} T_1^i = u_i \\ T_{n+1}^i = u_i + R_1^i + \ldots + R_{T_n^i}^i, \qquad \text{for } n \geq 1. \end{cases}$$

Note that the generation points are defined separately for each population and the detailed specification of the order of selection is left open. It will be seen that the observations between generation points

$$(2) \qquad \begin{cases} Z_0^i = u_i \\ Z_n^i = T_{n+1}^i - T_n^i, \qquad \text{for } n \geq 1, \end{cases}$$

form the generations of independent Galton-Watson branching processes for $i = A,B$, regardless of how the samples are ordered.

The sampling scheme with random sampling order within each generation may be visualized as an urn model:

Two urns are given; Urn I, a sampling urn, and Urn II, a holding urn. Initially, u_A balls of type A and u_B balls of type B are placed in the sampling urn. To begin the first generation of sampling, a ball is drawn at random from the sampling urn and its type noted. An observation is then made

on the population indicated and the response, $R \geq 1$ is recorded. A total of R balls of that type are then placed in the holding urn. The process is repeated until the sampling urn is empty. That is the end of the first generation of sampling. To begin the second generation, all the balls in the holding urn are placed in the sampling urn and sampling begins anew. The analysis to follow is based on the ball populations at those points where the sampling urn becomes empty. They are the generation points, T_n.

Theorem 1: Assume $m_A > m_B$. Then as n tends to infinity, the fractions of inferior selections

$$(3) \quad \begin{cases} \dfrac{T_n^B}{T_n^A + T_n^B}, \\[2ex] \dfrac{Z_n^B}{Z_n^A + Z_n^B} \end{cases}$$

approach zero with probability one.

Proof. Temporarily suppressing the population superscript i for A,B let $Z_n = T_{n+1} - T_n$ be the nth stage sample size. Then Z_n may be expressed as $X_{1,n} + \ldots + X_{Z_{n-1},n}$ where $X_{k,n} \equiv R_{T_{n-1}+k}$ are iid with R_1 and independent of Z_{n-1}, $n \geq 1$. Thus Z_n represents the nth generation of a Galton-Watson branching process initiated by $Z_0 = u$ ancestors and having offspring distribution equal to that of R_1 (Harris (1963)). The generation points $T_{n+1} = Z_0 + \ldots + Z_n$ are the cumulative progeny up to the nth generation. It follows that the expected generation size is

$$(4) \qquad EZ_n = \begin{cases} um^n & \text{if} \quad m < \infty \\ \infty & \text{if} \quad m = \infty \end{cases}, \quad m = ER_1$$

and the average sample number (ASN) is

$$(5)\qquad ASN = ET = \begin{cases} nu & \text{if } m = 1 \\ u\,\dfrac{m^n-1}{m-1} & \text{if } m < \infty \text{ and } m > 1 \\ \infty & \text{if } m = \infty \end{cases}$$

To prove (3), we first note that by assumption $m_A > m_B \geq 1$, we can choose an integer M so big that

$$(6)\qquad m_\star \equiv ER_n^A I_{[R_n^A \leq M]} > m_B.$$

Let $R_n^\star = R_n^A I_{[R_n^A \leq M]}$ and define $T_n^\star, Z_n^\star$ in a similar manner with $u_\star = u_A$.

Then for all $n \geq 1$

$$(7)\qquad T_n^A \geq T_n^\star.$$

Next, it is well-known that $\{\frac{Z_n^\star}{m_\star^n}, n\geq 1\}$ and $\{\frac{Z_n^B}{m_B^n}, n\geq 1\}$ are martingales.

Since $E\,\frac{Z_n^\star}{m_\star^n} = u_A < \infty$ and $E\,\frac{Z_n^B}{m_B^n} = u_B < \infty$, by the martingale convergence theorem, $\frac{Z_n^\star}{m_\star^n}$ and $\frac{Z_n^B}{m_B^n}$ converge with probability one to random variables $W^\star$ and W^B respectively as n tends to infinity. Furthermore, since $R^\star \leq M$, $W^\star$ is a strictly positive random variable with probability one. In view of

$$(8)\qquad \frac{T_{n+1}^\star}{m_\star^{n+1}} = \frac{1}{m_\star}\sum_{j=0}^{n} m_\star^{-(n-j)}\, Z_{n-j}^\star / m_\star^j,$$

$T_n^\star / m_\star^n$ converges to $W^\star/(m_\star-1)$ with probability one as n tends to infinity. Similarly if $m_B > 1$, T_n^B / m^n converges to $W^B/(m_B-1)$ with probability one and if $m_B = 1$, $T_n^B = nu_B$. Next, choose λ such that $\lambda\, \varepsilon(m_B, m_\star)$; then by (7)

$$(9) \qquad \frac{T_n^B}{T_n^A + T_n^B} \leq \frac{T_n^B}{T_n^* + T_n^B}$$

$$= \frac{\left(\frac{m_B}{\lambda}\right)^n \frac{T_n^B}{m_B^n}}{\left(\frac{m_*}{\lambda}\right)^n \frac{T_n^*}{m_*^n} + \left(\frac{m_B}{\lambda}\right)^n \frac{T_n^B}{m_B^n}}$$

which goes to zero with probability one as n tends to infinity. This also implies that $Z_n^B = o(T_{n+1}^A) = o(T_n^A + Z_n^A)$ with probability one since

$$(10) \qquad \frac{T_{n+1}^B}{T_{n+1}^A + T_{n+1}^B} \geq \frac{Z_n^B}{T_{n+1}^A + Z_n^B} .$$

As a result, it follows that

$$(11) \qquad \frac{Z_n^B}{Z_n^A + Z_n^B} = \frac{Z_n^B}{T_{n+1}^A - T_n^A + Z_n^B}$$

converges to zero with probability one as n tends to infinity.

The following corollaries show that favorable comparisons need not be restricted to the same generation points on the two populations.

<u>Corollary 1</u>: If $m_A^{d_A} > m_B^{d_B}$ for some positive integers d_A and d_B, then

$$(12) \qquad \left\{ \begin{array}{l} \dfrac{T_{nd_B}^B}{T_{nd_A}^A + T_{nd_B}^B} \\[2em] \dfrac{T_{nd_B}^B - T_{(n-1)d_B}^B}{T_{nd_A}^A - T_{(n-1)d_A}^A + T_{nd_B}^B - T_{(n-1)d_B}^B} \end{array} \right.$$

converge to zero with probability one as n tends to infinity. In particular if $m_B < m_A = \infty$, then for any positive integer d,

$$(13)\quad \begin{cases} \dfrac{T^B_{nd}}{T^A_n + T^B_{nd}} \\[2ex] \dfrac{T^B_{(n+1)d} - T^B_{nd}}{Z^A_n + T^B_{(n+1)d} - T^B_{nd}} \end{cases}$$

converge to zero with probability one as n tends to infinity.

Corollary 2: If $m_A^{d_A} > m_B^{d_B}$ for some positive integers d_A and d_B, then as n tends to infinity

$$(14)\quad \begin{cases} T^B_{nd_B}/T^A_{nd_A} = 0((m_B^{d_B}/m_A^{d_A})^n) \\[2ex] (T^B_{(n+1)d_B} - T^B_{nd_B})/(T^A_{(n+1)d_A} - T^A_{nd_A}) = 0((m_B^{d_B}/m_A^{d_A})^n) \end{cases}$$

with probability one.

Corollary 3: If $m_A > m_B$, then for any $p > 0$, as $m \to \infty$

$$(15)\quad \begin{cases} E(\dfrac{T^B_n}{T^A_n + T^B_n})^p \to 0, \\[2ex] E(\dfrac{Z^B_n}{Z^A_n + Z^B_n})^p \to 0. \end{cases}$$

If $m_A^{d_A} > m_B^{d_B}$ for some positive integers d_A and d_B, then for any $p > 0$, as $n \to \infty$

$$(16)\qquad \begin{cases} E\left(\dfrac{T^B_{nd_B}}{T^A_{nd_A}+T^B_{nd_B}}\right)^p \to 0, \\[2ex] E\left(\dfrac{T^B_{(n+1)d_B}-T^B_{nd_B}}{T^A_{(n+1)d_A}-T^A_{nd_A}+T^B_{(n+1)d_B}-T^B_{nd_B}}\right) \to 0 \ . \end{cases}$$

If $\infty = m_A > m_B$, then for any positive integer d and any $p > 0$, as $n \to \infty$

$$(17)\qquad \begin{cases} E\left(\dfrac{T^B_{nd}}{T^A_n+T^B_{nd}}\right)^p \to 0. \\[2ex] E\left(\dfrac{T^B_{(n+1)d}-T^B_{nd}}{Z^A_n+T^B_{(n+1)d}-T^B_{nd}}\right)^p \to 0. \end{cases}$$

Remark 1. It may be expected that $E(T^B_n/T^A_n)^p \to 0$ under general conditions. However we have only been able to prove it in the very special case of success/failure trials (see Section 4, Theorem 6).

Remark 2. The independence of $\{R^A_1, R^A_2, \ldots\}$ and $\{R^B_1, R^B_2, \ldots\}$ is not necessary for the results in Theorem 1 and Corollaries 1,2 and 3 to hold. The results also include the cases when either $m_A = \infty$ or $m_B = 1$ or both.

Remark 3. If there are K processes $\{R^A_1, R^A_2, \ldots\}. \ldots, \{R^K_1, R^K_2, \ldots\}$ with means $m_A, \ldots, m_K$ respectively, and if $m_A > \max(m_B, \ldots, m_K)$, then the results in Theorem 1 and Corollaries 1,2 and 3 will still hold when T^B and Z^B are replaced by $T^B + T^B + \ldots + T^K$ and $Z^B + Z^C + \ldots + Z^K$ respectively, and the conditions in the corollaries are changed from that of $m_A^{d_A} > m_B^{d_B}$ to that of $m_A^{d_A} > \max(m_B^{d_B}, \ldots, m_K^{d_K})$.

Section 3. Estimation in a Monotonic Branching Process

In this section we shall study the estimation of some of the parameters of the separate populations, specifically the mean responses m_i and the variances σ_i^2, $i=A,B$. As each separate population follows a Galton-Watson branching process, we shall suppress the superscript and subscript i. Let $R, R_1, \ldots$ be iid positive integer-valued random variables with $ER = m$. Let u be a positive integer and

$$T_0 = 0, \quad Z_0 = u \quad \text{and} \quad T_1 = u. \tag{18}$$

For $n \geq 1$, define

$$\begin{aligned} Z_n &= R_{T_{n-1}+1} + \ldots + R_{T_n} \\ T_{n+1} &= u + R_1 + \ldots + R_{T_n} \\ &= Z_0 + \ldots + Z_n. \end{aligned} \tag{19}$$

Notice that $Z_0 \leq Z_1 \leq Z_2 \ldots$, and for this reason we shall call this Galton-Watson branching process a monotonic branching process. For each $n \geq 0$, let F_n be the σ-field generated by $\{Z_0, \ldots, Z_n\}$. For the mean m, we shall consider the following two estimators

$$\begin{aligned} \hat{m}_n &= \frac{T_{n+1}-u}{T_n}, \\ \bar{m}_n &= \frac{Z_{n+1}}{Z_n}. \end{aligned} \tag{20}$$

These two estimators are well-known in the literature. See Dion and Keiding (1978) and the references therein. The estimator $\hat{m}_n$ is a maximum likelihood estimator of m. The following fact concerning the strong consistency of $\hat{m}_n$ and $\bar{m}_n$ is well-known.

Fact 1. Assume $m < \infty$. Then with probability one, as $n \to \infty$,

$$\text{(21)} \qquad \begin{aligned} &\text{(i)} \quad \hat{m}_n \to m, \\ &\text{(ii)} \quad \bar{m}_n \to m. \end{aligned}$$

In the following, we shall study the L_p-consistency of $\hat{m}_n$ and $\bar{m}_n$.

Definition. $\hat{\theta}_n$ is an L_p-consistent estimator of θ for some $p \geq 1$ if as $n \to \infty$.

$$\text{(22)} \qquad E|(\hat{\theta}_n - \theta)|^p \to 0.$$

To establish the L_p-consistency, we first develop a few results which are interesting in their own right.

Theorem 2: Assume that $ER^p < \infty$ for some $p \geq 1$. Let $a = \max(\frac{1}{2}, \frac{1}{p})$. Then

$$\text{(23)} \qquad \{|Z_n^{1-a}(\frac{Z_{n+1}}{Z_n} - m)|^p, \; n \geq 1\} \quad \text{is uniformly integrable.}$$

Proof. We decompose, for some K,

$$\text{(24)} \qquad \begin{aligned} R_n - m &= (R_n I_{[R_n \leq K]} - ER_n I_{[R_n \leq K]}) + (R_n I_{[R_n > K]} - ER_n I_{[R_n > K]}) \\ &\equiv X_n + Y_n, \quad \text{say.} \end{aligned}$$

Since $ER^p < \infty$, for all $\varepsilon > 0$, we can choose K so that

$$\text{(25)} \qquad E|Y_n|^p < \varepsilon.$$

First, let $s > \max(2,p)$. Then by the Marcinkiewicz-Zygmund inequality (see, e.g. Chow and Teicher (1978), p. 356), for some constant B_s,

$$(26) \qquad E\left|\frac{X_{T_n+1}+\dots+X_{T_{n+1}}}{Z_n^a}\right|^s$$

$$= EE\left(\frac{1}{Z^{as}}|X_{T_n+1}+\dots+X_{T_{n+1}}|^s|F_n\right)$$

$$\le B_s^s\, E\frac{1}{Z_n^{as}}\, E(|X^2_{T_n+1}+\dots+X^2_{T_{n+1}}|^{s/2}|F_n)$$

$$\le B_s^s\, E\frac{Z_n^{s/2-1}}{Z_n^{as}}\, E(|X_{T_n+1}|^s+\dots+|X_{T_{n+1}}|^s|F_n)$$

$$\le B_s^s K^s E\, Z_n^{s(\frac{1}{2}-a)} \le B_s^s K^s < \infty .$$

Therefore

$$(27) \qquad \left\{\left|\frac{X_{T_n}+\dots+X_{T_{n+1}}}{Z_n^a}\right|^p,\ n \ge 1\right\} \text{ is uniformly integrable.}$$

Next, consider the Y's. By the same Marcinkiewicz-Zygmund inequality, we have

$$(28) \qquad E\left|\frac{Y_{T_n+1}+\dots+Y_{T_{n+1}}}{Z_n^a}\right|^p$$

$$\le B_p^p\, E\,\frac{1}{Z_n^{ap}}\, E((Y^2_{T_n+1}+\dots+Y^2_{T_{n+1}})^{p/2}|F_n)$$

$$\le \begin{cases} B_p^p E\left(\frac{1}{Z_n^{ap}}\, E(|Y_{T_n+1}|^p+\dots+|Y_{T_{n+1}}|^p|F_n)\right), & \text{if } 1 \le p \le 2, \\ B_p^p\, E\left(\frac{Z_n^{p/2-1}}{Z_n^{ap}}\, E(|Y_{T_n+1}|^p+\dots+|Y_{T_{n+1}}|^p|F\ )\right), & \text{if } p > 2, \end{cases}$$

$$\le B_p^p\, \varepsilon\, E\,\frac{Z_n}{Z_n} = \varepsilon\, B_p^p,$$

which can be made arbitrarily small. Therefore

$$(29) \quad \left\{ \left|\frac{Y_{T_n+1}+\dots+Y_{T_{n+1}}}{Z_n^a}\right|^p, \ n \geq 1 \right\} \text{ is uniformly integrable.}$$

Combining (27) and (29), we have the desired result (23).

Corollary 4: If $ER^p < \infty$ for some $p \geq 1$, then

(30)

(i) $\{|\bar{m}_n - m|^p, n \geq 1\}$ is uniformly integrable.

(ii) $\{|\hat{m}_n - m|^p, n \geq 1\}$ is uniformly integrable.

Proof. (i) Since $|\bar{m}_n - m|^p \leq |Z_n^{1-a}(\frac{Z_{n+1}}{Z_n} - m)|^p$, ($Z_n \geq 1$ and $a \leq 1$), the result follows from Theorem 2.

(ii) By (i), $\{|\bar{m}_n|^p, n \geq 1\}$ is uniformly integrable. We note that

$$(31) \quad \frac{T_{n+1}}{T_n} = \frac{T_n+Z_n}{T_n} = 1 + \frac{Z_n}{T_n} \leq 1 + \frac{Z_n}{Z_{n-1}} .$$

Therefore $\{(\frac{T_{n+1}}{T_n})^p, n \geq 1\}$ is uniformly integrable and it follows that

$$(32) \quad \{|\frac{T_{n+1}-u}{T_n} - m|^p, n \geq 1\} \text{ is uniformly integrable.}$$

Corollary 5: Assume $ER^p < \infty$ for some $p \geq 1$. Then as $n \to \infty$

(33)

(i) $E|\bar{m} - m|^p \to 0$, L_p-consistency,

(ii) $E|\hat{m} - m|^p \to 0$, L_p-consistency.

Proof. The result follows from Fact 1 and Corollary 4.

Corollary 6: If $ER^p < \infty$ for some $p \geq 2$ and $\sigma^2 = \operatorname{Var} R \in (0,\infty)$, then as $n \to \infty$,

$$(34) \quad E|Z_n^{\frac{1}{2}}(\frac{Z_{n+1}}{Z_n} - m)|^p \to \sigma^p \int_{-\infty}^{\infty}|x|^p \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx,$$

and for any positive odd integer $j \leq p$,

$$(35) \quad E(Z_n^{\frac{1}{2}}(\frac{Z_{n+1}}{Z_n} - m))^j \to 0.$$

Proof. Since as $n \to \infty$

$$(36) \quad Z_n^{\frac{1}{2}}(\frac{Z_{n+1}}{Z_n} - m) \xrightarrow{D} N(0,\sigma^2)$$

(See e.g. Nagaev (1967) or Dion (1974), the results follow from Theorem 2.

Next we shall establish a result analogous to Theorem 2 for T_n. We need the following elementary lemma.

Lemma 1: Let $V, V_1, \ldots$ be iid positive random variables with $P[V \geq 1] = 1$ and $EV = m > 1$. Then for all $q \geq 1$, there is an α with $\alpha \in (0,1)$ such that

$$(37) \qquad \sup_{n \geq 1} E(\frac{n}{V_1+\ldots+V_n})^q \leq \alpha.$$

Proof. By the strong law of large numbers, as $n \to \infty$

$$(38) \qquad \frac{n}{V_1+\ldots+V_n} \to \frac{1}{m} < 1$$

with probability one. Since $(n/(V_1+\ldots+V_n))^q \leq 1$, as $n \to \infty$

$$(39) \qquad E(\frac{n}{V_1+\ldots+V_n})^q \to \frac{1}{m^q} .$$

by the bounded convergence theorem. Therefore there is an $\alpha^* < 1$ and an integer n_0 such that for all $n \geq n_0$

$$(40) \quad E(\frac{n}{V_1+\ldots+V_n})^q \leq \alpha^*.$$

Since $E(n/V_1+\ldots+V_n))^q < 1$ for $n = 1,\ldots,n_0$, take

$$(41) \quad \alpha = \max(\alpha^*, E(\frac{1}{V_1})^q, \ldots, E(\frac{n_0}{V_1+\ldots+V_{n_0}})^q)$$

and the result follows.

Theorem 3: Assume that $ER^p < \infty$ for some $p > 1$. Let $a = \max(\frac{1}{2},\frac{1}{p})$. Then for any $r < p$

$$\left\{\left|\frac{T_{n+1}-u-mT_n}{Z_{n-1}^a}\right|^r, \ n \geq 1\right\} \text{ is uniformly integrable.} \tag{42}$$

Proof. Let r be less than p. Choose s such that $s \geq 1$ and $r < s < p$. Then by Minkowski and Holder inequalities

$$\left(E\left|\frac{T_{n+1}-u-mT_n}{Z_{n-1}^a}\right|^s\right)^{1/s} \tag{43}$$

$$\leq \sum_{k=1}^{n}\left(E\left(\frac{Z_{k-1}}{Z_{n-1}}\right)^{as}\left|\frac{Z_k-mZ_{k-1}}{Z_{k-1}^a}\right|^s\right)^{1/s}$$

$$\leq \sum_{k=1}^{n}\left(E\left|\frac{Z_{k-1}}{Z_{n-1}}\right|^{\frac{asp}{p-s}}\right)^{\frac{p-s}{sp}}\left(E\left|\frac{Z_k-mZ_{k-1}}{Z_{k-1}^a}\right|^p\right)^{1/p}.$$

Let $q = asp/(p-s)$. Then by Lemma 1, for some $0 < \alpha < 1$,

$$E(\frac{Z_{k-1}}{Z_{n-1}})^q = E(E((\frac{Z_{k-1}}{Z_{n-1}}\frac{Z_{n-2}}{Z_{n-2}})^q|F_{n-2}))$$

$$= E(\frac{Z_{k-1}}{Z_{n-2}})^q\ E((\frac{Z_{n-2}}{Z_{n-1}})^q|F_{n-2}) \tag{44}$$

$$\leq \alpha\ E(\frac{Z_{k-1}}{Z_{n-2}})^q \leq \alpha^{n-k}\ .$$

By Theorem 2, $\sup_{k\geq 1}(E|\frac{Z_k-mZ_{k-1}}{Z_{k-1}^a}|^p)^{\frac{1}{p}} \leq C$ for some finite constant C. Therefore from (43) and (44), we have

$$(45)\qquad \left(E\left|\frac{T_{n+1}-u-mT_n}{Z_{n-1}^a}\right|^s\right)^{1/s} \le C\sum_{k=1}^{n}(\alpha^{n-k})^{\frac{p-s}{sp}} = C\,\frac{1-\alpha^{\frac{n(p-s)}{sp}}}{1-\alpha^{\frac{p-s}{sp}}} \le \frac{C}{1-\alpha^{\frac{p-s}{sp}}}$$

and the result follows.

Corollary 7: If $ER^p<\infty$ for some $p>1$ and $a = \max(\frac{1}{2},\frac{1}{p})$, then for any $r<p$,

$$(46)\qquad \{|T_n^{1-a}(\frac{T_{n+1}-u}{T_n} - m)|^r,\ n \ge 1\} \text{ is uniformly integrable.}$$

Proof. In view of

$$(47)\qquad |T_n^{1-a}(\frac{T_{n+1}-u}{T_n} - m)| \le |\frac{T_{n+1}-u-mT_n}{Z_{n-1}^a}|,$$

the result follows from Theorem 3.

Corollary 8: If $ER^p < \infty$ for some $p \ge 2$ and $\mathrm{Var}\, R = \sigma^2 \varepsilon(0,\infty)$, then for any $r < p$, as $n \to \infty$

$$(48)\qquad E|T_n^{\frac{1}{2}}(\frac{T_{n+1}-u}{T_n} - m)|^r \to \sigma^r \int_{-\infty}^{\infty} \frac{|x|^r e^{-x^2/2}}{\sqrt{2\pi}}\, dx$$

and for any positive odd integer $j < p$,

$$(49)\qquad E(T^{\frac{1}{2}}(\frac{T_{n+1}-u}{T_n} - m))^j \to 0.$$

Proof. Since as $n \to \infty$,

$$T_n^{\frac{1}{2}}(\frac{T_{n+1}-u}{T_n} - m) \xrightarrow{D} N(0,\sigma^2),$$

(see e.g. Dion (1974) or Jagers (1975)); the result follows from Corollary 7.

Remark 4. We conjecture that the r in Theorem 3 and Corollaries 7 and 8

can be improved to p, in which case $p \geq 1$ can replace $p > 1$ in Theorem 3 and Corollary 7; and $j < p$ can be changed to $j \leq p$ in Corollary 8.

In the remainder of this section we shall assume that $\mathrm{Var}\, R = \sigma^2$ which is finite and positive. We shall study the following estimators of σ^2:

$$\text{i)}\quad \bar{\sigma}_n^2 = \frac{1}{n}\sum_{k=1}^{n} Z_{k-1}\left(\frac{Z_k}{Z_{k-1}} - \frac{Z_n}{Z_{n-1}}\right)^2,$$

(50)

$$\text{(ii)}\quad \tilde{\sigma}_n^2 = \frac{1}{n}\sum_{k=1}^{n} Z_{k-1}\left(\frac{Z_k}{Z_{k-1}} - m\right)^2.$$

The consistency of these estimators is given in the literature and we collect them into the following fact.

Fact 2.

(i) Heyde (1974). As $n \to \infty$, $\bar{\sigma}_n^2 \to \sigma^2$ with probability one.

(51)

(ii) Dion (1975). As $n \to \infty$, $\tilde{\sigma}_n^2 \to \sigma^2$ in probability.

In the following, we shall study the L_p-consistency of these two estimators of σ^2.

Theorem 4: Assume that $ER^{2p} < \infty$ for some $p \geq 1$ and let $a = \max(\frac{1}{2},\frac{1}{p})$. Then

(52) $\{|n^{1-a}(\tilde{\sigma}_n^2 - \sigma^2)|^p,\ n \geq 1\}$ is uniformly integrable.

Proof. For each $k \geq 1$,

$$(53)\qquad E\left(Z_{k-1}\left(\frac{Z_k}{Z_{k-1}} - m\right)^2 - \sigma^2 \,\middle|\, F_{k-1}\right)$$

$$= \frac{1}{Z_{k-1}} E((Z_k - mZ_{k-1})^2 | F_{k-1}) - \sigma^2 = 0.$$

Since $ER^{2p} < \infty$, Theorem 2 implies the uniform integrability of $\{(Z_{k-1}(\frac{Z_k}{Z_{k-1}} - m)^2)^p,\ k \geq 1\}$. By Lemmas 1 and 2 in Chow and Yu (1984),

$$(54)\quad \{|\frac{1}{n^a}\sum_{k=1}^{n}(Z_{k-1}(\frac{Z_k}{Z_{k-1}}-m)^2-\sigma^2)|^p,\ n\geq 1\}\ \text{is uniformly integrable,}$$

which is the desired result.

Theorem 5: Assume that $ER^{2p}<\infty$ for some $p\geq 1$, and let $a=\max(\frac{1}{2},\frac{1}{p})$. Then

$$(55)\qquad \{|n^{1-a}(\bar{\sigma}_n^2-\sigma^2)|^p,\ n\geq 1\}\ \text{is uniformly integrable.}$$

Proof. We decompose

$$(56)\quad n^{1-a}(\bar{\sigma}_n^2-\sigma^2)$$

$$=\frac{1}{n^a}\{\sum_{k=1}^{n}(Z_{k-1}(\frac{Z_k}{Z_{k-1}}-m)^2-\sigma^2)+(m-\frac{Z_n}{Z_{n-1}})^2\sum_{k=1}^{n}Z_{k-1}$$

$$-2\sum_{k=1}^{n}(Z_k-Z_{k-1}m)(\frac{Z_n-mZ_{n-1}}{Z_{n-1}})\}.$$

In view of Theorem 4, it suffices to show the uniform integrability of

$$\text{(i)}\quad \{|\frac{1}{n^a}(m-\frac{Z_n}{Z_{n-1}})^2\sum_{k=1}Z_{k-1}|^p,\ n\geq 1\}\ \text{and}$$

$$\text{(ii)}\quad \{|\frac{1}{n^a}\sum_{k=1}^{n}(Z_k-mZ_{k-1})(\frac{Z_n-mZ_{n-1}}{Z_{n-1}})|^p,\ n\geq 1\}.$$

For (i), by Theorem 2 (for some finite constant C) and by Lemma 1 (for some $\alpha\ \varepsilon\ (0,1)$) and (44) we have

$$(58)\quad E(\frac{1}{n^{ap}}((\frac{mZ_{n-1}-Z_n}{Z_{n-1}^{1/2}})^2)^p(\sum_{k=1}^{n}\frac{Z_{k-1}}{Z_{n-1}})^p)$$

$$=E\,\frac{1}{n^{ap}}(\sum_{k=1}^{n}\frac{Z_{k-1}}{Z_{n-1}})^p\,E(|\frac{mZ_{n-1}-Z_n}{Z_{n-1}^{1/2}}|^{2p}|F_{n-1})$$

$$\leq\frac{C}{n^{ap}}E(\sum_{1}^{n}\frac{Z_{k-1}}{Z_{n-1}})^p\leq\frac{C}{n^{ap}}(\sum_{1}^{n}(E(\frac{Z_{k-1}}{Z_{n-1}})^p)^{1/p})^p$$

$$\le \frac{C}{n^{ap}} \left(\sum_{1}^{n} \alpha^{(n-k)/p}\right)^{p} \le \frac{C}{n^{ap}} \frac{1}{(1-\alpha^{1/p})^{p}} \to 0, \text{ as } n \to \infty$$

yielding (i). For (ii), by the same results used for (i), we have

$$(59) \quad \frac{1}{n^{ap}} E\left|\sum_{k=1}^{n} \frac{(Z_k - mZ_{k-1})}{\sqrt{Z_{k-1}}} \frac{(Z_n - mZ_{n-1})}{\sqrt{Z_{n-1}}} \sqrt{\frac{Z_{k-1}}{Z_{n-1}}}\right|^{p}$$

$$\le \frac{1}{n^{ap}} \left(\sum_{k=1}^{n} \left(E\left|\frac{Z_k - mZ_{k-1}}{\sqrt{Z_{k-1}}}\right|^{p} \left|\frac{Z_n - mZ_{n-1}}{\sqrt{Z_{n-1}}}\right|^{p} \left(\frac{Z_{k-1}}{Z_{n-1}}\right)^{p/2}\right)^{1/p}\right)^{p}$$

$$\le \frac{1}{n^{ap}} \left(\sum_{1}^{n} \left(E\left|\frac{Z_k - mZ_{k-1}}{\sqrt{Z_{k-1}}}\right|^{2p}\right)^{1/2p} \left(E\left|\frac{Z_n - mZ_{n-1}}{\sqrt{Z_{n-1}}}\right|^{2p} \left(\frac{Z_{k-1}}{Z_{n-1}}\right)^{p}\right)^{1/2p}\right)^{p}$$

$$\le \frac{C^{\frac{1}{2}}}{n^{ap}} C^{\frac{1}{2}} \left(\sum_{1}^{n} \left(E\left(\frac{Z_{k-1}}{Z_{n-1}}\right)^{p}\right)^{1/2p}\right)^{p} \le \frac{C}{n^{ap}} \frac{1}{(1-\alpha^{1/2p})^{p}} \to 0 \quad \text{as} \quad n \to \infty$$

yielding (ii). And this completes the proof.

Corollary 9: *If* $ER^{2p} < \infty$ *for some* $p \ge 1$, *then*

(60)
(i) $\{|\tilde{\sigma}_n^2 - \sigma^2|^p, n \ge 1\}$ is uniformly integrable

(ii) $\{|\bar{\sigma}_n^2 - \sigma^2|^p, n \ge 1\}$ is uniformly integrable.

Proof. Let σ_n^2 be either $\tilde{\sigma}_n^2$ or $\bar{\sigma}_n^2$. Since $a \le 1$,
$n^{1-a}|\sigma_n^2 - \sigma^2| \ge |\sigma_n^2 - \sigma^2|$ and the result follows from Theorems 4 and 5.

Corollary 10: If $ER^{2p} < \infty$ for some $p \ge 1$, then as $n \to \infty$

(61)
(i) $E|\tilde{\sigma}_n^2 - \sigma^2|^p \to 0$, L_p-consistency,

(ii) $E|\bar{\sigma}_n^2 - \sigma^2|^p \to 0$, L_p-consistency.

Proof. By Fact 2, $\tilde{\sigma}_n^2$ and $\bar{\sigma}_n^2$ converge to σ^2 in probability as $n \to \infty$. Together with this, Corollary 9 gives the L_p-consistency.

Corollary 11: If $ER^{2p} < \infty$ for some $1 \leq p < 2$, then as $n \to \infty$

$$(62) \quad \begin{aligned} &\text{(i)} \quad E|n^{\frac{p-1}{p}}(\tilde{\sigma}_n^2 - \sigma^2)|^p \to 0, \\ &\text{(ii)} \quad E|n^{\frac{p-1}{p}}(\bar{\sigma}_n^2 - \sigma^2)|^p \to 0. \end{aligned}$$

Proof. By Lemma 1 of Chow and Yu (1984) and (53), as $n \to \infty$

$$(63) \quad E|n^{\frac{p-1}{p}}(\tilde{\sigma}_n^2 - \sigma^2)|^p \to 0.$$

By (56), (58), (59) and (63),

$$E|n^{\frac{p-1}{p}}(\bar{\sigma}_n^2 - \sigma^2)|^p \to 0.$$

Corollary 12: If $ER^{2p} < \infty$ for some $p \geq 2$, then as $n \to \infty$

$$(64) \quad E|n^{\frac{1}{2}}(\bar{\sigma}_n^2 - \sigma^2)|^p \to 2^{\frac{p}{2}} \sigma^{2p} \int_{-\infty}^{\infty} \frac{|x| \, e^{-x^2/2}}{\sqrt{2\pi}} \, dx;$$

and for any positive odd integer $j \leq p$,

$$(65) \quad E(n^{\frac{1}{2}}(\bar{\sigma}_n^2 - \sigma^2))^j \to 0.$$

Consequently as $n \to \infty$

$$(66) \quad E(\bar{\sigma}_n^2) = \sigma^2 + o(\frac{1}{\sqrt{n}}),$$

i.e. the bias is of smaller order than $n^{-\frac{1}{2}}$, and

$$(67) \quad \mathrm{Var}(\bar{\sigma}_n^2) = \frac{2\sigma^2}{n} + o(\frac{1}{n}).$$

Proof. Heyde (1974) has shown that as $n \to \infty$

$$(68) \quad n^{\frac{1}{2}}(\bar{\sigma}_n^2 - \sigma^2) \xrightarrow{D} N(0, 2\sigma^4).$$

By Theorem 5, the results follow.

Section 4. Iterative Play-the-Winner Sampling

As a natural extension of the Zelen (1969) approach to binary comparisons, let R_1^i equal to the number of trials on treatment i, $i = A$ or B, until the first failure is observed. So the initial generation does not correspond to any set of trials but only to the number of success runs to be observed in the first generation of sampling on treatment i. The responses, R_k^i, have a geometric distribution with finite mean $m_i = 1/q_i$ and finite variance $\sigma_i^2 = (1-q_i)/q_i$ where q_i is the probability of failure. The ASN is $u_i(1-q_i^n)/q_i^n(1-q_i)$ over the first $T_{n+1}^i - u_i$ trials corresponding to generations 1 through n. Note that at least nu_i trials are run on treatment i, but there is no absolute upper bound on the actual number of trials for any $n \geq 1$.

The estimator of $p_i = 1 - q_i$, $\hat{p}_i = 1 - 1/m_i$, reduces to $(Z_n^i - u_i)/(T_{n+1}^i - u_i)$ which equals the cumulative number of successes divided by the number of treatments on treatment i in this scheme. The fraction of inferior treatment selections is small if n is large almost surely and in the L_p sense. An approximate distribution is available. Since $W_i (= \lim_{n\to\infty} Z_n^i/m_i^n)$ has a gamma distribution with moment generating function $E(e^{sW_i}) = (1 + s/u_i)^{-u_i}$, $s > 0$, (Harris (1963)), $2u_iW_i$ has a chi-squared distribution with $2u_i$ degrees of freedom. Thus the ratio, $C_A(n)T_n^A/C_B(n)T_n^B$, has an approximate F distribution with $2u_A$ and $2u_B$ degrees of freedom, where the constants are $C_i(n) = p_i(q_i^{n-1})/(1 - q_i^{n-1})$, $i = A, B$. Furthermore, by Corollary 2, $T_n^B/T_n^A = O((q_A/q_B)^n)$ with probability one, as $n \to \infty$.

Theorem 6 shows that L_p convergence obtains as well.

<u>Theorem 6</u>: Assume that $1/q_A = m_A > m_B = 1/q_B$. If $P[R_1^i = x] = (1-q_i)^{x-1}q_i$, $x = 1,2,\ldots$, $i = A,B$, then for any $p > 0$, as $n \to \infty$

(i) $$E\left(\frac{Z_n^B}{Z_n^A}\right)^p \to 0,$$

(ii) $$E\left(\frac{T_n^B}{T_n^A}\right)^p \to 0.$$

<u>Proof</u>. Without loss of generality, assume that $u_A = u_B = 1$ and that p is an integer. Then

$$E(Z_n^B)^p = \sum_{x=1}^{\infty} x^p(1-q_B^n)^{x-1}q_B^n$$

$$\leq \sum_{x=1}^{\infty} x(x+1)\ldots(x+p-1)(1-q_B^n)^{x-1}q_B^n = p!\ q_B^{-np}$$

$$E\left(\frac{1}{Z_n^A}\right)^p = \sum_{x=1}^{\infty} \frac{1}{x^p}(1-q_A^n)^{x-1}q_A^n$$

$$= \sum_{x=1}^{p} \frac{1}{x^p}(1-q_A^n)^{x-1}q_A^n + \sum_{x=p+1}^{\infty} \frac{1}{x^p}(1-q_A^n)^{x-1}q_A^n$$

$$\leq pq_A^n + \sum_{p+1}^{\infty} \frac{1}{x(x-1)\ldots(x-p+1)}(1-q_A^n)^{x-1}q_A^n$$

$$\leq pq_A^n - \frac{Cnq_A^{np}\log q_A}{1-q_A^n}, \quad \text{for some constant } C.$$

Hence as $n \to \infty$

(i) $$E\left(\frac{Z_n^B}{Z_n^A}\right)^p = E(Z_n^B)^p\, E\left(\frac{1}{Z_n^A}\right)^p \to 0.$$

Next

$$E(T_{n+1}^B)^p$$

$$\leq ((E(Z_0^B)^p)^{1/p}+\dots+(E(Z_n^B)^p)^{1/p})^p$$

$$\leq (1 + (p!q_B^{-p})^{1/p}+\dots+(p!q_B^{-np})^{1/p})^p$$

$$\leq p!(1 + q_B^{-1}+\dots+ q_B^{-n})$$

$$= p!\left(\frac{q_B^{-n-1}-1}{q_B^{-1}-1}\right) \leq \frac{p!}{(q_B^{-1}-1)}\, q_B^{-(n+1)p}$$

(ii) $$E\left(\frac{T_{n+1}^B}{T_{n+1}^A}\right)^p \leq E(T_{n+1}^B)^p\, E\left(\frac{1}{Z_n^A}\right)^p$$

$$\leq \frac{p!}{(q_B^{-1}-1)^p}\, \frac{1}{q_B^{(n+1)p}}\, \left(pq_A^n - \frac{Cnq_A^{np}\log q_A}{1-q_A^n}\right) \to 0 \quad \text{as} \quad n \to \infty.$$

Corollary 13: Assume that $q_A^{-d_A} = m_A^{d_A} > m_B^{d_B} = q_B^{-d_B}$ for some positive integers d_A and d_B,

(i) $E(Z_{nd_A}^B/Z_{nd_A}^A)^p \to 0$ as $n \to \infty$

(ii) $E(T_{nd_A}^B/T_{nd_A}^A)^p \to 0$ as $n \to \infty$

Remark 5. The delicacy of the above result is noteworthy. It seems plausible that a similar convergence obtains in non-geometric cases but no proof is known.

Remark 6. Certainly other selection procedures exist for the present case, Bather (1981). However direct comparisons are difficult because the sample sizes are random in regenerative sampling but are fixed in Bather's study.

The actual implementation of the trials is only mildly constrained by

by the requirement that the generation points T_n^A, T_n^B be reached on both treatments for some fixed value of n. It is interesting that if it is assumed, in addition, that every generation point $T_1^i,\ldots,T_n^i$ up to the nth be reached on both treatments $i = A,B$ before proceeding to the next, then the present scheme can be visualized as an adaptive iteration of the Zelen scheme. In particular, let $(Z_1^A, Z_1^B) = PW(u_A, u_B)$ denote the number of trials on the two treatments in a Zelen type experiment which is stopped when u_A failures are observed on treatment A and u_B failures are observed on treatment B. Then the successive sample sizes are generated recursively by $(Z_1^A, Z_1^B) = PW(u_A, u_B)$, $(Z_k^A, Z_k^B) = PW(Z_{k-1}^A, Z_{k-1}^B)$, $k = 2,\ldots,n$. The urn scheme presented in Section 2 can be adapted to the present case in which the total response is spread over a number of trials. As before, select a ball from the sampling urn and note its type. Administer the indicated treatment and observe the response. If it is a success a ball of the corresponding type is added to the holding urn and the selected ball is returned to the sampling urn. If a failure is observed then the selected ball is simply transferred to the holding urn. As before, a generation is complete when the sampling urn becomes empty.

In view of Fact 1 in Section 3 as applied to the present iterative play-the-winner design, it is plausible to base a test of hypothesis about the failure rates q_i on the number of successes observed over n_i sampling generations, $i = A,B$. In particular, for positive integers n_A, n_B, a test is proposed for

$$\text{(69)} \qquad H_0\colon q_A^{n_A} \geq q_B^{n_B} \quad \text{vs} \quad H : q_A^{n_A} < q_B^{n_B}$$

based on the conditional distribution of $S^A = Z_{n_A}^A - u_A$ given $S = S^A + S^B$,

where $S^B = Z^B_{n_B} - u$. The random variables S^A and S^B are independent and represent the number of successes on treatments A and B over trials $1,2,\ldots,T^A_n$ and $1,2,\ldots,T^B_n$ respectively. if $n_A = n_B = 1$ and $u_A = u_B = f$, the test is equivalent to that of Zelen (1969).

Since the samples on the two treatments are independent and since the geometric distribution is preserved under the composition of probability generating functions in the Galton-Watson branching process (Harris(1963)), S^A, S^B and S have negative binomial distributions and the conditional distribution can be presented explicitly. For r a non-negative integer and $k = 0,1,\ldots,r$,

$$(70)\quad g(k|r) \equiv P[S = k|S^A = r]$$
$$= \binom{k+u_A-1}{u_A-1}\binom{r-k+u_B-1}{u_B-1}\lambda^k \Big/ \sum_{j=0}^{r} \binom{j+u_A-1}{u_A-1}\binom{r-j+u_B-1}{u_B-1}\lambda^j,$$

where $\lambda = (1-q_A^{n_A})(1-q_B^{n_B})$.

Under $q_A^{n_A} = q_B^{n_B}$, $\lambda = 1$ and the distribution with r a nonnegative integer and $x = 0,1,\ldots,r$, is

$$(71)\qquad G(x|r) \equiv P[S^A \leq x|S = r] = \sum_{k=0}^{x} g(k|r).$$

The α-level test is proposed:

(72) Reject H_0 in favor of H_1 if and only if $G(S^A|S) > 1-\alpha$.

In the simplest case, $u_A = u_B = 1$, the power can be bounded as follows:

<u>Theorem 7</u>: Assume that $u_A = u_B = 1$, n_A and n_B are positive integers and $0 < \alpha \leq \frac{1}{2}$. Then

$$(73) \quad (q^{-n_B} + (1-q_B^{-n_B})(1-q_A^{n_A})^{\theta})^{-1} \le K_\alpha \equiv P[G(S^A|S) > 1-\alpha]$$

$$\le (1-q^{n_A})(q^{-n_B} + (1-q^{-n_B})(1-q^{n_A})^{\theta})^{-1},$$

where $\theta = \alpha^{-1} - 1$ and $K_\alpha = P[G(S^A|S) > 1-\alpha]$.

7 Remark 5. The lower bound is exact if $\alpha = ½$, and for α small, approximately

$$(74) \qquad q_B^{n_B} \le K_\alpha \le q_B^{n_B} / (1-q_A^{n_A}).$$

Proof. Since $u_A = u_B = 1$, $G(x|r) = (x+1)/(r+1)$, $x = 0,1,\ldots,r$. Thus $K_\alpha = P[\theta(S^A+1) > S^B]$. Since

$$(75) \qquad P[\text{Reject } H_0|S^B = k] = P[S^A + 1 > \theta k]$$

$$= (1-q^{n_A})^{[\theta k]},$$

and the latter is bounded below by $(1-q^{n_A})^{\theta k}$ and above by $(1-q^{n_A})^{\theta k-1}$, K_α is seen to satisfy the bound stated upon averaging over the values of S^B.

In view of the interesting outcome of the clinical study by Bartlett et al (1985), Cornell et al (1986), performance values with q_A close to zero are presented in the Table. Of course, the power is conserved upon truncation of the favored treatment since the success counts are cumulative. So the null hypothesis could still be rejected without ever completing a generation on the better treatment. If it appears that $q_A = 0$ may be true, as in the aforementioned study, then the trial may be concluded at any point after the specified generation point is reached on treatment B. Of course at least $u_B n_B$ trials shall be run on treatment B with regenerative sampling.

TABLE

Iterative Play-the-Winner

$n_A = u_A = u_B = 1, \quad q_A = .04$ and $q_B = .80$

$H_0: q_A \geq q_B^{n_B}$ vs $H_1: q_A < q_B^{n_B}$

ASNA = 25

n_B	$q_B^{n_B}$	ASNB	K_α $\alpha=.01$		$\alpha=.05$		$\alpha=.50$
			LB	UB	LB	UB	
1	.80	1.25	.80	.84	.88	.92	.99
2	.64	2.81	.64	.67	.76	.80	.98
3	.512	4.77	.52	.54	.66	.69	.96
5	.328	10.25	.33	.35	.47	.49	.92
8	.168	24.80	.17	.18	.27	.28	.83

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A regenerative sampling plan is proposed for the sequential comparison of two populations having positive integral response. It is designed to be both an extension and an improvement of the play-the-winner rules for binary trials in the sense that a much wider variety of responses is allowed, the fraction of inferior selections approaches zero, and the play-the-winner rule is contained as a special case. Almost sure convergence and moment convergence in the pth order is studied for the fraction of inferior selections and for a maximum likelihood estimator of the mean response. A conditional test of hypothesis is given for the binary case.

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